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ENERGY DISSIPATION OF A THIN ELASTOPLASTIC TUBE UNDER TORSION AND COMPRESSION

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Abstract—An analytical computation is performed of the energy dissipated by a thin tube made of a work-hardening material, subjected to both a static compressive load and a dynamic torsional load.

1. INTRODUCTION

The issue of energy dissipation in elastoplastic materials is interesting from both a theoretical and practical point of view. On the theoretical side it is important to test the existing theories of dissipation (Coleman and Owen, 1977; Šilhavý, 1980a,b) in order to develop a consistent thermodynamic theory for elastoplastic materials (Lucchesi, 1993). On the practical side, dissipation is a crucial feature, for instance, in the design of antiseismic dampers.

Despite its importance, there have been to our knowledge no reports of explicit calculations of the energy dissipated from a thin tube under conditions of loading and unloading in stress regimes that are more complex than the monoaxial case.

In this paper we present an analytical computation of a homogeneous, infinitesimal deformation of a thin tube, subjected to both a normal strain that is constant in time, due to a permanent load, and a torsional strain varying in time with a prescribed law. The hypothesis that the strain is everywhere the same considerably simplifies the explicit computation of the energy dissipated, because the solution of the equilibrium problem reduces to solving the system of ordinary differential equations that describes the constitutive response. The explicit solution of this system is immediate in the case of monoaxial stress, which is the one almost exclusively examined in applications (Su *et al.*, 1989). More complex stress regimes are also interesting from the point of view of applications, because the dissipating devices are entrusted with both the dynamic loads and part of the permanent loads, and such loads generally have different directions.

In the present paper the hypothesis is made that the material comprising the tube's wall is an initially isotropic von Mises elastoplastic material (Lucchesi *et al.*, 1992). For materials with combined work hardening, the explicit computation of the dissipated energy is laborious; therefore we carry out separate computations for isotropic and kinematic hardening mechanisms; then, we compare our results with the case of an ideally plastic material. Finally, as an example, a more detailed analysis is made of the situation in which the torsional strain varies in time according to a sine law.

2. CONSTITUTIVE RELATIONSHIPS

The materials we deal with are well-known elastoplastic materials with a von Mises yield criterion, associated flow rule and isotropic and kinematic hardening. In this section we summarize some constitutive relationships, written in the strain space according to our pourposes. We refer the reader to Lucchesi *et al.* (1992) for a detailed exposition of these results. We begin with a few notations.

We use Sym to denote the space of all the symmetric tensors of the second order, and $\mathbf{I} \in \mathrm{Sym}$ to indicate the identity tensor. For each $\mathbf{A} \in \mathrm{Sym}$, $\mathbf{A}_0 = \mathbf{A} - (1/3\mathrm{tr}\,\mathbf{A})\mathbf{I}$, with $\mathrm{tr}\,\mathbf{A}$ the trace of $\mathbf{A}$, denotes the deviator of $\mathbf{A}$; Sym_0 is the subspace of Sym made up of all the deviatoric tensors. For $\mathbf{A}, \mathbf{B} \in \mathrm{Sym}$, $\mathbf{A} \cdot \mathbf{B} = \mathrm{tr}(\mathbf{AB})$ is the inner product of $\mathbf{A}$ and $\mathbf{B}$; $\|\mathbf{A}\| = (\mathbf{A} \cdot \mathbf{A})^{1/2}$ is the norm of $\mathbf{A}$.

A *deformation process* is a continuous and piecewise continuously differentiable mapping, $\mathbf{E}: [0, \bar{\tau}] \to \mathrm{Sym}$, with $\mathbf{E}(0) = 0$, whose value at a typical instant τ is interpreted as the current infinitesimal strain, i.e. the symmetric part of the displacement gradient, measured with respect to a fixed reference configuration and at a fixed material point. For each instant $\tau \in [0, \bar{\tau}]$,

$$E(\tau) = \{\mathbf{A} \in \mathrm{Sym} \mid \|\mathbf{A}_0 - \mathbf{C}(\tau)\| \leqslant \hat{\rho}(\tau)\}, \quad (1)$$

with $\mathbf{C}(\tau) \in \mathrm{Sym}_0$, is the *elastic range* corresponding to the deformation process $\mathbf{E}$. $E(\tau)$ is a cylinder in Sym, whose base $E(\tau)_0 = \{\mathbf{A}_0 \in \mathrm{Sym}_0 \mid \|\mathbf{A}_0 - \mathbf{C}(\tau)\| \leqslant \hat{\rho}(\tau)\}$ is the ball of Sym_0 with centre $\mathbf{C}(\tau)$ and radius $\hat{\rho}(\tau)$. It is assumed that $E(\tau)$ contains the current deformation $\mathbf{E}(\tau)$; its points are interpreted as all the infinitesimal deformations of the reference configuration that are elastically attainable starting from the current configuration.

The *plastic deformation process* corresponding to $\mathbf{E}$ is a mapping $\mathbf{E}^p: [0, \bar{\tau}] \to \mathrm{Sym}_0$ that at each instant τ delivers the value $\mathbf{E}^p(\tau)$ of the (unique) deformation belonging to $E(\tau)_0$ to which a null stress corresponds.

The *Odqvist parameter*

$$\zeta(\tau) = \int_0^\tau \|\dot{\mathbf{E}}^p(\tau')\| \,\mathrm{d}\tau' \quad (2)$$

measures the length of the plastic deformation process until instant τ. We accept the usual associated flow rule:

$$\dot{\mathbf{E}}^p(\tau) = \dot{\zeta}(\tau)\mathbf{N}(\tau), \quad \mathbf{E}(\tau)_0 \in \partial E(\tau)_0, \quad (3)$$

where

$$\mathbf{N}(\tau) = \hat{\rho}(\tau)^{-1}(\mathbf{E}(\tau)_0 - \mathbf{C}(\tau)) \quad (4)$$

is the outward unit tensor to $\partial E(\tau)_0$ at $\mathbf{E}(\tau)_0$.

For each $\tau \in [0, \bar{\tau}]$, the stress at instant τ, during the deformation process $\mathbf{E}$ is given by

$$\mathbf{T}(\tau) = \hat{\mathbf{E}}(\mathbf{E}(\tau) - \mathbf{E}^p(\tau)) = 2\mu(\mathbf{E}(\tau) - \mathbf{E}^p(\tau)) + \lambda(\mathrm{tr}\,\mathbf{E}(\tau))\mathbf{I}, \quad (5)$$

with $\hat{\mathbf{E}}$ the elasticity tensor and λ, μ the Lame' moduli of the material. From eqn (5) we obtain

$$\mathbf{T}(\tau)_0 = 2\mu(\mathbf{E}(\tau)_0 - \mathbf{E}^p(\tau)); \quad \mathrm{tr}\,\mathbf{T}(\tau) = 3\chi\,\mathrm{tr}\,\mathbf{E}(\tau), \quad (6)$$

where $\chi = 1/3(2\mu + 3\lambda)$ is the bulk modulus.

Finally we accept the following hardening rules. There exist three material constants, ρ_0, β and η, with

$$\rho_0 > 0, \quad \beta \geqslant 0, \quad \eta \geqslant 0, \tag{7}$$

such that

$$\hat{\rho}(\tau) = \rho(\zeta(\tau)) = \rho_0 + \beta\zeta(\tau) \quad \text{(isotropic hardening rule)}, \tag{8}$$

$$\mathbf{C}(\tau) = (1+\eta)\mathbf{E}^p(\tau) \quad \text{(kinematic hardening rule)}. \tag{9}$$

In particular, if both $\beta = 0$ and $\eta = 0$, the material is said to be *ideally plastic*.

The evolution of the plastic deformation is governed by the following relationship (Lucchesi *et al.*, 1992):

$$\dot{\zeta}(\tau) = \begin{cases} 0 \text{ if } \|\mathbf{E}(\tau)_0 - \mathbf{C}(\tau)\| < \rho(\zeta(\tau)) \\ 0 \text{ if } \|\mathbf{E}(\tau)_0 - \mathbf{C}(\tau)\| = \rho(\zeta(\tau)) \text{ and } \mathbf{N}(\tau)\cdot\dot{\mathbf{E}}(\tau)_0 \leqslant 0 \\ (1+\eta+\beta)^{-1}\mathbf{N}(\tau)\cdot\dot{\mathbf{E}}(\tau)_0 \\ \text{if } \|\mathbf{E}(\tau)_0 - \mathbf{C}(\tau)\| = \rho(\zeta(\tau)) \text{ and } \mathbf{N}(\tau)\cdot\dot{\mathbf{E}}(\tau)_0 > 0 \end{cases}. \tag{10a–c}$$

The third condition in eqn (10) is called the *plastic loading condition*. Relationships (3), (4) and (10) can be conveniently written as a single equation. Indeed, if we put

$$\mathbf{X}(\tau) = (\mathbf{E}(\tau)_0 - \mathbf{C}(\tau)), \tag{11}$$

$$\kappa = (1+\eta+\beta)^{-1}(1+\eta), \tag{12}$$

we get

$$\dot{\mathbf{X}}(\tau) = \begin{cases} \dot{\mathbf{E}}(\tau)_0 \text{ if } \|\mathbf{X}(\tau)\| < \rho(\zeta(\tau)) \\ \dot{\mathbf{E}}(\tau)_0 \text{ if } \|\mathbf{X}(\tau)\| = \rho(\zeta(\tau)) \text{ and } \mathbf{X}(\tau)\cdot\dot{\mathbf{E}}(\tau)_0 \leqslant 0 \\ \dot{\mathbf{E}}(\tau)_0 - \kappa\rho^{-2}(\zeta(\tau))(\mathbf{X}(\tau)\cdot\dot{\mathbf{E}}(\tau)_0)\mathbf{X}(\tau) \\ \text{if } \|\mathbf{X}(\tau)\| = \rho(\zeta(\tau)) \text{ and } \mathbf{X}(\tau)\cdot\dot{\mathbf{E}}(\tau)_0 > 0. \end{cases} \tag{13a–c}$$

Moreover, since the material is supposed to be initially isotropic, we have that $\mathbf{E}^p(0) = 0$, so therefore

$$\mathbf{X}(0) = 0. \tag{14}$$

In the applications considered hereafter, the differential equation (13) with the initial condition (14) has one, and only one, solution; its integration makes it possible to determine $\mathbf{E}^p$, with the help of eqns (9) and (11), and the stress can then be calculated by means of eqn (6). For each deformation process $\mathbf{E}$, let

$$w[\tau_1, \tau_2] = \int_{\tau_1}^{\tau_2} \mathbf{T}(\tau)\cdot\dot{\mathbf{E}}(\tau)\,d\tau \tag{15}$$

be the work per unit volume done by internal forces from instant τ_1 to instant τ_2. It is known that (Lucchesi, 1993)

$$w[\tau_1, \tau_2] = \tfrac{1}{2}\{(\mathbf{E}(\tau_2) - \mathbf{E}^p(\tau_2))\cdot\hat{\mathbf{E}}[\mathbf{E}(\tau_2) - \mathbf{E}^p(\tau_2)] - (\mathbf{E}(\tau_1) - \mathbf{E}^p(\tau_1))\cdot\hat{\mathbf{E}}[\mathbf{E}(\tau_1) - \mathbf{E}^p(\tau_1)]\} + \mu\eta(\|\mathbf{E}^p(\tau_2)\|^2 - \|\mathbf{E}^p(\tau_1)\|^2) + 2\mu\{\omega(\zeta(\tau_2)) - \omega(\zeta(\tau_1))\}, \tag{16}$$

where

$$\omega'(\zeta) = \rho(\zeta) = \rho_0 + \beta\zeta. \tag{17}$$

In particular, if $\mathbf{E}(\tau_2) = \mathbf{E}(\tau_1)$ and $\mathbf{E}^p(\tau_2) = \mathbf{E}^p(\tau_1)$, it can be deduced from eqns (16) and (17) that

$$w[\tau_1, \tau_2] = 2\mu\{\rho_0(\zeta(\tau_2) - \zeta(\tau_1)) + \tfrac{1}{2}\beta(\zeta(\tau_2)^2 - \zeta(\tau_1)^2)\}. \tag{18}$$

3. FORMULATION OF THE PROBLEM

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal base of R^3. Let us consider a thin tube T (Fig. 1) with its generatrix parallel to $\mathbf{e}_1$, constrained in such a way that, at the points of base B_1, displacements both in the direction of the tangent to ∂B_1 and in that of $\mathbf{e}_1$ are prevented. The hypothesis is made that the tube is subjected to a given constant load $f\mathbf{e}_1$, uniformly distributed on the base B_2, as well as to a torsional strain varying in time in a prescribed manner. Therefore, since the deformation is supposed to be homogeneous, in each point $\mathbf{p}$ of the wall of T with the outward normal parallel to $\mathbf{e}_3$ the deviatoric deformation process is

$$\mathbf{E}(\tau)_0 = e_{11}(\tau)\mathbf{A} + \varphi(\tau)\mathbf{B}, \tag{19}$$

where

$$\mathbf{A} = \mathbf{e}_1 \otimes \mathbf{e}_1 - \tfrac{1}{2}(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3), \quad \mathbf{B} = (\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) \tag{20}$$

and $\tau \to \varphi(\tau)$ is a function assigned in the interval $[0, \bar{\tau}]$ which is supposed to be continuously differentiable and null for $\tau = 0$. From eqns (19) and (5) and from considerations on the equilibrium of T, the expression of the stress

$$\mathbf{T}(\tau) = \mathbf{f}(\mathbf{e}_1 \otimes \mathbf{e}_1) + \sigma_{12}(\tau)(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \tag{21}$$

can be deduced and therefore, from eqns (19), (21) and (6), for the deformation process we obtain

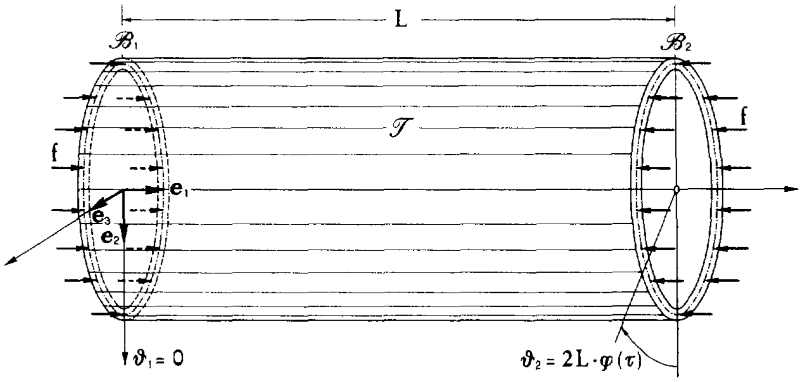

Fig. 1. Thin tube subjected to constant load $f\mathbf{e}_1$ and torsonal strain φ.

$$\mathbf{E}(\tau) = \varepsilon_{11}(\tau)\mathbf{e}_1 \otimes \mathbf{e}_1 + \tfrac{1}{2}((3\chi)^{-1}f - \varepsilon_{11}(\tau))(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3) + \varphi(\tau)(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \quad (22)$$

where

$$\varepsilon_{11}(\tau) = e_{11}(\tau) + \tfrac{1}{3}\operatorname{tr}\mathbf{E} = e_{11}(\tau) + (9\chi)^{-1}f. \quad (23)$$

Let us suppose that, for $\tau \in [0, \tau_1^*]$, $\tau \to \varphi(\tau)$ is an increasing function and let $\tau_1 \in [0, \tau_1^*]$ be the instant of first yielding, i.e., the maximum value of τ for which we get

$$\| E(\tau)_0 \| \leqslant \rho_0. \quad (24)$$

For $\tau \leqslant \tau_1$ we have, in view of eqns (19), (21) and (6),

$$e_{11}(\tau) = \tfrac{1}{3}(f/\mu), \quad \sigma_{12}(\tau) = 2\mu\varphi(\tau) \quad (25)$$

and therefore τ_1 can be obtained from the equation

$$\varphi(\tau_1) = \sqrt{\tfrac{1}{2}\rho_0^2 - \tfrac{1}{3}(f/2\mu)^2}, \quad (26)$$

a consequence of eqns (19) and (23).

For $\tau \in [\tau_1, \tau_1^*]$ the plastic loading condition is verified. In order to calculate the Odqvist parameter, let us begin by observing that, in view of eqns (3), (10) and (19), we can write

$$\mathbf{E}^p(\tau) = e^p_{11}(\tau)\mathbf{A} + e^p_{12}(\tau)\mathbf{B}; \quad (27)$$

moreover, from eqns (19) and (27) we get

$$\dot{\mathbf{E}}(\tau)_0 = \dot{e}_{11}(\tau)\mathbf{A} + \dot{\varphi}(\tau)\mathbf{B}, \quad \dot{\mathbf{E}}^p(\tau) = \dot{e}^p_{11}(\tau)\mathbf{A} + \dot{e}^p_{12}\mathbf{B}. \quad (28)$$

Therefore, taking into account eqn (11), there exist two functions $\tau \mapsto \chi_1(\tau)$ and $\tau \mapsto \chi_2(\tau)$, such that

$$\mathbf{X}(\tau) = \chi_1(\tau)\mathbf{A} + \chi_2(\tau)\mathbf{B}. \quad (29)$$

Since we have

$$\|\mathbf{A}\|^2 = \tfrac{3}{2}, \quad \|\mathbf{B}\|^2 = 2, \quad \mathbf{A}\cdot\mathbf{B} = 0, \quad (30)$$

from eqns (10c), (28) and (29) we obtain

$$\dot{\zeta} = \rho^{-1}(1+\eta+\beta)^{-1}(\tfrac{3}{2}\dot{e}_{11}\chi_1 + 2\dot{\varphi}\chi_2), \quad (31)$$

and from eqns (13c), and (21)

$$\dot{e}_{11} - \kappa\rho^{-2}(\tfrac{3}{2}\dot{e}_{11}\chi_1^2 + 2\dot{\varphi}\chi_1\chi_2) = \dot{\chi}_1, \quad (32)$$

$$\dot{\varphi} - \kappa\rho^{-2}(\tfrac{3}{2}\dot{e}_{11}\chi_1\chi_2 + 2\dot{\varphi}\chi_2^2) = \dot{\chi}_2. \quad (33)$$

In view of eqns (21) and (6), we have

$$e_{11}(\tau) - e^{p}_{11}(\tau) = \tfrac{1}{3}(f/\mu); \tag{34}$$

therefore, since $(e_{11}(\tau) - e^{p}_{11}(\tau))$ is constant, from eqns (11), (9), (19) and (27) we obtain

$$\dot{\chi}_1 = -\eta \dot{e}_{11}. \tag{35}$$

From eqn (32), bearing in mind eqn (35) and the plastic loading condition

$$\rho^2 = \tfrac{3}{2}\chi_1^2 + 2\chi_2^2, \tag{36}$$

we can deduce

$$\dot{e}_{11} = 2\chi_1\chi_2((\eta+\beta)\rho^2 + 2\chi_2^2)^{-1}\dot{\varphi}. \tag{37}$$

With the help of eqns (36) and (37), by setting

$$\bar{\chi}_2 = \rho^{-1}\chi_2, \tag{38}$$

eqn (31) becomes

$$\dot{\zeta} = 2\bar{\chi}_2(2\bar{\chi}_2^2 + \eta + \beta)^{-1}\dot{\varphi}. \tag{39}$$

By differentiating eqn (36) with respect to τ, we get

$$2\rho\beta\dot{\zeta} = 3\chi_1\dot{\chi}_1 + 4\chi_2\dot{\chi}_2 \tag{40}$$

and, with the help of this, from eqn (33) we get

$$\dot{\bar{\chi}}_2 = \rho^{-1}(1 + 2(\kappa/\eta)\bar{\chi}_2^2)^{-1}\{(1 - 2\kappa\bar{\chi}_2^2)\dot{\varphi} + (\beta/\eta)(\kappa - \eta - 2\kappa\bar{\chi}_2^2)\bar{\chi}_2\dot{\zeta}\}. \tag{41}$$

Substituting eqn (39) into eqn (41), we have

$$\dot{\bar{\chi}}_2 = \rho^{-1}(\eta + 2\kappa\bar{\chi}_2^2)^{-1}\{\eta(1 - 2\kappa\bar{\chi}_2^2) + 2\beta\bar{\chi}_2^2(\eta + \beta + 2\bar{\chi}_2^2)^{-1}(\kappa - \eta - 2\kappa\bar{\chi}_2^2)\}\dot{\varphi}. \tag{42}$$

Moreover, from eqns (10), (24), (11), (8), (9) and (38) we can deduce

$$\zeta(\tau_1) = 0, \quad \bar{\chi}_2(\tau_1) = \rho_0^{-1}\varphi(\tau_1). \tag{43}$$

Equations (39) and (42), together with the initial conditions (43), constitute a system of differential equations, the integration of which makes it possible to calculate ζ and $\bar{\chi}_2$ in the interval $[\tau_1, \tau_1^*]$; $\rho(\zeta)$, χ_1 and χ_2 can then be calculated from eqns (8), (36) and (38) and, in their turn, these make it easy to determine **E**, $\mathbf{E}^p$ and **T**. In the case that is more interesting in applications, i.e. that in which φ is not a monotonous function throughout the interval $[0, \bar{\tau}]$, the computation procedure described in this section can be applied to each interval $[\tau_k, \tau_k^*]$ in which the plastic loading condition is verified, provided the initial conditions (43) are substituted by the following:

$$\zeta(\tau_k) = 0, \quad \bar{\chi}_2(\tau_k) = \rho_0^{-1}[\varphi(\tau_k) - (1+\eta)e^{p}_{12}(\tau^*_{k-1})]. \tag{44a,b}$$

4. KINEMATIC HARDENING

Let us suppose we get

$$\eta > 0, \quad \beta = 0. \tag{45a,b}$$

Then, for $\tau \in [\tau_k, \tau_k^*]$, the differential equations (39) and (42) that rule the problem, become

$$\dot{\zeta} = 2\bar{\chi}_2(\eta + 2\bar{\chi}_2^2)^{-1}\dot{\varphi}, \tag{46}$$

$$\dot{\bar{\chi}}_2 = \eta\rho_0^{-1}(\eta + 2\bar{\chi}_2^2)^{-1}(1 - 2\bar{\chi}_2^2)\dot{\varphi}, \tag{47}$$

from which, eliminating $\dot{\varphi}$ and integrating, we obtain

$$\zeta(\tau) = -(\rho_0/2\eta)\ln(1 - 2\bar{\chi}_2(\tau)^2) + c_1, \tag{48}$$

where

$$c_1 = (\rho_0/2\eta)\ln(1 - 2\bar{\chi}_2(\tau_k)^2) + \zeta(\tau_k). \tag{49}$$

Obtaining $\bar{\chi}_2$ from eqn (48) and substituting in eqn (46), we have

$$\dot{\zeta} = \sqrt{2}g(\zeta)/(g^2(\zeta) + \eta)\dot{\varphi}, \tag{50}$$

where

$$g(\zeta) = \sqrt{1 - \exp(-2\eta\rho_0^{-1}(\zeta - c_1))} = \sqrt{1 - \alpha\exp[-2\eta\rho_0^{-1}(\zeta - \zeta(\tau_k))]} \tag{51}$$

with

$$\alpha = [1 - 2\bar{\chi}_2(\tau_k)^2]. \tag{52}$$

Integrating this last equation, with the initial condition $(44)_1$, we obtain (Brychkov *et al.*, 1989)

$$2g(\zeta) + (1+\eta)\ln\{(1+g(\zeta))^{-1}(1-g(\zeta))\} = -4\eta\sqrt{2}\rho_0^{-1}(\varphi(\tau) - \varphi(\tau_k)). \tag{53}$$

Equation (53) constitutes an implicit link between ζ and τ. On the other hand, in many applications the kinematic hardening parameter η is much smaller than unity (Bennati and Lucchesi, 1991) and this can be used to obtain an approximate expression of ζ as a function of τ. Indeed, if we put

$$\psi(\eta) = \sqrt{1 - \alpha\exp[-2\eta\rho_0^{-1}(\zeta - \zeta(\tau_k))]}, \tag{54}$$

with a direct computation, we obtain

$$\psi(0) = \sqrt{1-\alpha}, \tag{55}$$

$$\psi'(0) = \alpha(\zeta - \zeta(\tau_k))(\rho_0\sqrt{1-\alpha})^{-1}, \tag{56}$$

$$\psi''(0) = -\alpha(2-\alpha)(\zeta - \zeta(\tau_k))^2(\rho_0^2(1-\alpha)\sqrt{1-\alpha})^{-1}. \tag{57}$$

Similarly, given

$$\bar{\psi}(\eta) = \ln\{(1+\psi(\eta))^{-1}(1-\psi(\eta))\}, \tag{58}$$

we have

$$\bar{\psi}(0) = \ln\{(1+\sqrt{1-\alpha})^{-1}(1-\sqrt{1-\alpha})\}, \tag{59}$$

$$\bar{\psi}'(0) = -2(\zeta-\zeta(\tau_k))(\rho_0\sqrt{1-\alpha})^{-1}, \tag{60}$$

$$\bar{\psi}''(0) = 2\alpha(\zeta-\zeta(\tau_k))^2(\rho_0^2(1-\alpha)\sqrt{1-\alpha})^{-1}. \tag{61}$$

In view of eqns (54)–(61), it can be shown from eqn (53) that,

$$\tfrac{1}{2}\alpha\eta\rho_0^{-1}(\zeta(\tau)-\zeta(\tau_k))^2+(1-\alpha+\eta)(\zeta(\tau)-\zeta(\tau_k))-\sqrt{2(1-\alpha)}(\varphi(\tau)-\varphi(\tau_k)) \simeq 0, \tag{62}$$

where the sign $\simeq$ indicates that the relationship holds within an error of order $\sigma(\eta)$. From eqn (62) we obtain

$$\zeta(\tau) \simeq \zeta(\tau_k)+\zeta_p(\tau)h_k(\tau), \tag{63}$$

with

$$\zeta_p(\tau) = \rho_0\chi_2(\tau_k)^{-1}(\varphi(\tau)-\varphi(\tau_k)), \tag{64}$$

$$h_k(\tau) = \{1-\tfrac{1}{2}\eta\rho_0^2\chi_2(\tau_k)^{-2}(1+\tfrac{1}{2}\alpha\rho_0^{-1}\zeta_p(\tau))\}. \tag{65}$$

In order to calculate the internal work done in the interval $[\tau_k, \tau]$, it is also necessary to know $\mathbf{E}$ and $\mathbf{E}^p$. For this purpose, we observe that eqns (38) and (48) make it possible to write

$$\chi_2(\tau)^2 = \tfrac{1}{2}\rho_0^2[1-\alpha+2\alpha\rho_0^{-1}(\zeta(\tau)-\zeta(\tau_k))\eta-2\alpha\rho_0^{-2}(\zeta(\tau)-\zeta(\tau_k))^2\eta^2]+\sigma(\eta^2), \tag{66}$$

from which, in view of eqn (63), it can be shown that

$$\chi_2(\tau)^2 = \chi_2(\tau_k)^2+\alpha\rho_0\zeta_p\eta-\alpha\zeta_p^2[1+\rho_0/((1-\alpha)\zeta_p)+\alpha/(2(1-\alpha))]\eta^2+\sigma(\eta^2) \tag{67}$$

and therefore

$$\chi_2(\tau) \simeq \chi_2(\tau_k)[1+\alpha\zeta_p(\tau)\eta/((\rho_0(1-\alpha))]. \tag{68}$$

From eqns (11), (9), (28) and (29) we can deduce

$$e_{12}^p(\tau) = (\varphi(\tau)-\chi_2(\tau))(1+\eta)^{-1} \simeq (\varphi(\tau)-\chi_2(\tau))(1-\eta), \tag{69}$$

and therefore, in view of eqn (68),

$$e_{12}^p(\tau) \simeq \varphi(\tau)-\chi_2(\tau_k)-\eta[\varphi(\tau)-\chi_2(\tau_k)(1+\alpha\zeta_p(\tau)/((\rho_0(1-\alpha)))]. \tag{70}$$

Moreover, from eqns (36), (67) and (52) we can deduce that

$$\chi_1(\tau) = \chi_1(\tau_k)[1-(\zeta_p(\tau)/\rho_0)\eta+a(\tau)\eta^2]+\sigma(\eta^2), \tag{71}$$

where

$$a(\tau) = \zeta_p^2[\tfrac{2}{3}\alpha + \tfrac{2}{3}\rho_0\alpha/((1-\alpha)\zeta_p) + \tfrac{1}{3}\alpha^2/(1-\alpha) - 3\rho_0^2/2]. \tag{72}$$

Since, in view of eqns (11) and (34), we have

$$e_{11}^p(\tau) = \eta^{-1}(\tfrac{1}{3}f/\mu - \chi_1(\tau)), \tag{73}$$

from eqn (71) we obtain

$$e_{11}^p(\tau) \simeq e_{11}^p(\tau_k) + \chi_1(\tau_k)\zeta_p(\tau)/\rho_0 - a(\tau)\chi_1(\tau_k)\eta. \tag{74}$$

This equation makes it possible to calculate $e_{11}(\tau)$, with the help of eqn (34). In particular, if the material is ideally plastic, putting $\eta = 0$ in eqns (63), (68), (70), (71) and (74) we obtain, for $\tau \in [\tau_k, \tau_k^*]$,

$$\zeta(\tau) = \zeta_p(\tau), \tag{75}$$

$$\chi_2(\tau) = \chi_2(\tau_k), \tag{76}$$

$$e_{12}^p(\tau) = \varphi(\tau) - \chi_2(\tau_k), \tag{77}$$

$$\chi_1(\tau) = \tfrac{1}{3}(f/\mu), \tag{78}$$

$$e_{11}^p(\tau) = \tfrac{1}{3}(f/\mu)\chi_2(\tau_k)^{-1}(\varphi(\tau) - \varphi(\tau_k)) \tag{79}$$

and, from these last two equations, we can deduce that

$$e_{11}(\tau) = \tfrac{1}{3}(f/\mu)\chi_2(\tau_k)^{-1}\varphi(\tau). \tag{80}$$

Finally, when $\mathbf{E}(\tau)$, $\mathbf{E}^p(\tau)$ and $\zeta(\tau)$ are known, the internal work done in the interval $[\tau_k, \tau]$ can be calculated from eqn (16), bearing in mind that

$$\omega(\zeta) = \rho_0\zeta \tag{81}$$

follows from eqn (45b).

5. ISOTROPIC HARDENING

Let us suppose that we get

$$\eta = 0, \quad \beta > 0. \tag{82}$$

From eqns (39), (42) and (82) we can deduce, for $\tau \in [\tau_k, \tau_k^*]$,

$$\dot{\zeta} = 2\chi_2(\beta + 2\chi_2^2)^{-1}\dot{\varphi}, \tag{83}$$

$$\dot{\chi}_2 = \beta\rho^{-1}(\beta + 2\chi_2^2)^{-1}(1 - 2\chi_2^2)\dot{\varphi}, \tag{84}$$

from which

$$\beta\rho^{-1}\dot{\zeta} = 2\chi_2(1 - 2\chi_2^2)^{-1}\dot{\chi}_2. \tag{85}$$

Integrating eqn (85) in the interval $[\tau_k, \tau]$ we obtain

$$2\chi_2^2(\tau)+\alpha(\rho_0/\rho(\tau))^2 = 1, \tag{86}$$

where α is defined by eqn (52). From eqns (83), (86) and the relationship $2\beta\rho\dot{\zeta} = (\rho^2)^{\cdot}$ we obtain

$$\tfrac{1}{2}\rho^{-2}\beta^{-1}(\sqrt{2(\rho^2-\alpha\rho_0^2)})^{-1}(\rho^2(1+\beta)-\alpha\rho_0^2)(\rho^2)^{\cdot} = \dot{\varphi}. \tag{87}$$

Integrating this latter equation on interval $[\tau_k, \tau]$, we have (Brychkov *et al.*, 1989)

$$\sqrt{2}\beta(\varphi(\tau)-\varphi(\tau_k)) = (1+\beta)\sqrt{\rho(\zeta)^2-\alpha\rho_0^2}-\rho_0\sqrt{\alpha}\arctan(\sqrt{\rho(\zeta)^2/(\alpha\rho_0^2)-1}+c_2, \tag{88}$$

with

$$c_2 = (1+\beta)\rho_0\sqrt{1-\alpha}-\rho_0\sqrt{\alpha}\arctan(\sqrt{(1-\alpha)/\alpha}). \tag{89}$$

As in the case of kinematic hardening, an implicit relation between ζ and τ has been obtained. Observing that, in the applications, β is often much smaller than units, with a procedure similar to that used in the previous section we obtain

$$\sqrt{2}(\varphi(\tau)-\varphi(\tau_k)) \simeq \zeta(\tau)\sqrt{(1-\alpha)}+\tfrac{1}{2}\beta\zeta(\tau)(1+\alpha\zeta(\tau)/\rho_0)/\sqrt{(1-\alpha)}, \tag{90}$$

where the sign $\simeq$ indicates that the relationship holds within an error of order $\sigma(\beta)$. It can be shown from eqns (90) and (43) that

$$\zeta(\tau) \simeq \zeta_p(\tau)[1-\tfrac{1}{4}\beta\rho_0^2\chi_2(\tau_k)^{-2}(1+\sqrt{2}(\alpha/\rho_0)\zeta_\rho(\tau))], \tag{91}$$

ζ_p being defined by eqn (64).

The calculation of e_{12}^p can be carried out, observing that from eqns (86), (8) and (44b) it is easy to obtain

$$\chi_2(\tau)^2 = \chi_2(\tau_k)^2+\tfrac{1}{2}(\rho(\zeta(\tau))^2-\rho_0^2) \simeq \chi_2(\tau_k)^2+\beta\rho_0\zeta(\tau) \simeq \chi_2(\tau_k)^2+\beta\rho_0\zeta_\rho(\tau), \tag{92}$$

from which

$$\chi_2(\tau) \simeq \chi_2(\tau_k)(1+\tfrac{1}{2}\beta\rho_0\chi_2(\tau_k)^{-2}\zeta_\rho(\tau)) \tag{93}$$

which, in its turn, implies

$$e_{12}^p(\tau) = \varphi(\tau)-\chi_2(\tau) \simeq (\varphi(\tau)-\chi_2(\tau_k))(1-\tfrac{1}{2}\beta\chi_2(\tau_k)^{-2}\rho_0^2). \tag{94}$$

In order to calculate e_{11}^p, we observe that, in the case of isotropic hardening, $\chi_1(\tau) = e_{11}(\tau)-e_{11}^p(\tau)$ is constant. Then, from eqn (37), and in view of the results obtained in the present section, we can show that

$$\dot{e}_{11}^p \simeq \tfrac{1}{3}(f/\mu)\chi_2(\tau_k)^{-1}[1-\tfrac{1}{2}\beta\rho_0^2\chi_2(\tau_k)^{-2}(\zeta_p(\tau)/\rho_0+1)]\dot{\varphi}(\tau) \tag{95}$$

which, integrated over the interval $[\tau_k, \tau]$, gives

$$e_{11}^p(\tau) \simeq e_{11}^p(\tau_k)+\tfrac{1}{3}(f/\mu)\chi_2(\tau_k)^{-1}(\varphi(\tau)-\varphi(\tau_k))[1-\tfrac{1}{4}\beta\rho_0^2\chi_2(\tau_k)^{-3}(\varphi(\tau)+3\varphi(\tau_k)-2\chi_2(\tau_k))], \tag{96}$$

from which, with the help of eqn (34), $e_{11}(\tau)$ can be immediately calculated.

In this case, too, when $\mathbf{E}$, $\mathbf{E}^p$ and ζ are known, the internal work can be calculated from eqn (16); for this purpose it should be noted that, in view of eqns (17), (82) and (91), we have

$$\omega(\zeta(\tau)) \simeq \rho_0 \zeta_p(\tau) q_k(\tau), \tag{97}$$

where

$$q_k(\tau) = 1 - \tfrac{1}{4}\beta\rho_0^2\chi_2(\tau_k)^{-2}[1 + 2\zeta_p\rho_0^{-1}[\tfrac{1}{2}\sqrt{2} - (\sqrt{2}-1)\chi_2(\tau_k)^2\rho_0^{-2}]. \tag{98}$$

6. SINUSOIDAL DEFORMATION PROCESS

In this section we shall examine the case in which the behavior of the torsional strain to which the thin tube is subjected is sinusoidal (Fig. 2).

$$\varphi(\tau) = \xi \sin \tau, \tag{99}$$

with ξ an assigned constant.

First of all, we shall determine the intervals of time $[\tau_k, \tau_k^*]$ during which the plastic loading condition is verified. Since φ takes its extremal values at instants τ_k^*, we have

$$\tau_k^* = \tfrac{1}{2}\pi(2k-1), \quad k = 1, 2, 3, \ldots. \tag{100}$$

Moreover, since the behaviour of the material is elastic in the interval $[\tau_{k-1}^*, \tau_k]$, we get

$$\chi_2(\tau_k) = -\chi_2(\tau_{k-1}^*), \tag{101}$$

from which we obtain

$$\varphi(\tau_k) - (1+\eta)e_{12}^p(\tau_k) = -(\varphi(\tau_{k-1}^*) - (1+\eta)e_{12}^p(\tau_{k-1}^*)), \tag{102}$$

which, since $e_{12}^p(\tau_{k-1}^*) = e_{12}^p(\tau_k)$ is known, makes it possible to calculate the instant of kth yielding

$$\tau_k = \arcsin\{\xi^{-1}[2(1+\eta)e_{12}^p(\tau_{k-1}^*) + (-1)^{k+1}]\}. \tag{103}$$

We shall now go on to calculate the expression for the internal work in this load condition. First of all, let us suppose the material hardens kinematically so that eqn (45) holds. We see that in intervals $[\tau_k, \tau_k^*]$, in view of eqn (36), the point having coordinates

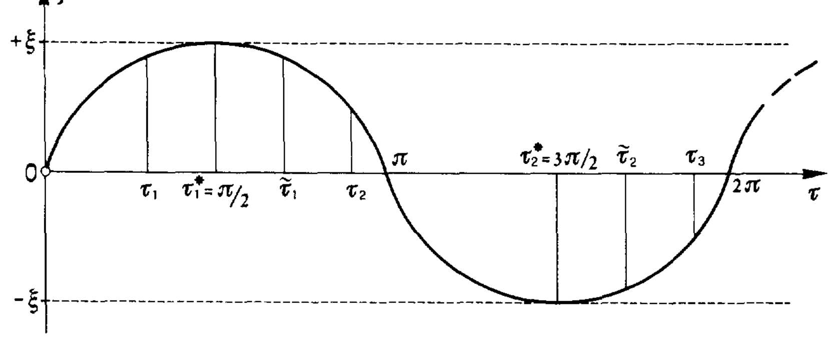

Fig. 2. Sinusoidal law of torsional strain.

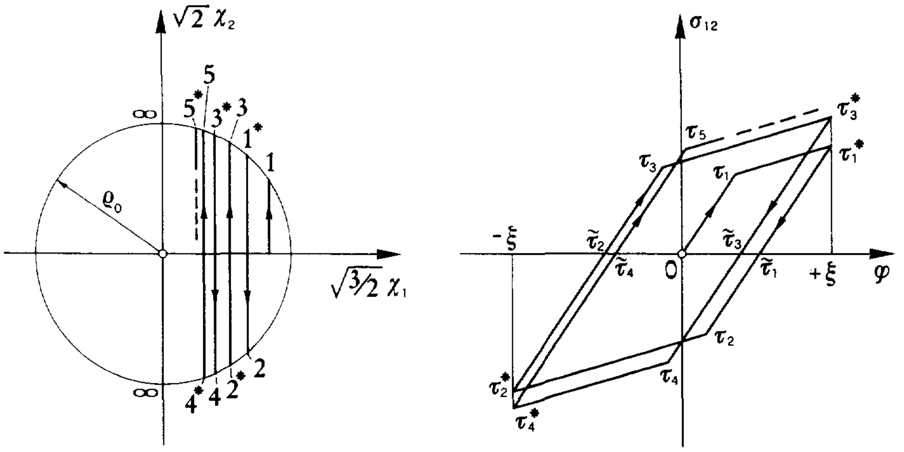

Fig. 3. Deformation process and hysteresis diagram for kinematic hardening.

$(\sqrt{\tfrac{3}{2}}\chi_1, \sqrt{2}\chi_2)$ belongs to the circumference with centre in the origin and radius ρ_0 (Fig. 3). Given that $\tilde{\tau}_k \in [\tau_k^*, \tau_{k+1}]$ expresses the instant at which σ_{12} is null, we deduce from eqn (16) that the internal work done after k oscillations is

$$w[0, \tilde{\tau}_k] = 2\mu\rho_0\zeta(\tau_k^*) + \mu\eta\|\mathbf{E}^p(\tau_k^*)\|^2. \tag{104}$$

With the help of eqn (63), we obtain the Odqvist parameter

$$\zeta(\tau_k^*) \simeq \sum_{n=1}^{k} \zeta_p(\tau_n^*)h_n(\tau_n^*), \tag{105}$$

where, in the light of eqns (99), (100) and (101), we have

$$\zeta_p(\tau_n^*) = \rho_0\xi|\chi_2(\tau_n)^{-1}|(1+(-1)^n \sin(\tau_n)). \tag{106}$$

Moreover, from eqn (68) we can deduce

$$\chi_2(\tau_k^*) = \chi_2(\tau_k)\{1+\tfrac{1}{2}\eta\rho_0^{-1}\chi_2(\tau_k)^{-2}(\rho_0^2-2\chi_2(\tau_k)^2)\zeta_p(\tau_k^*)\}, \tag{107}$$

from which, in its turn, we obtain $e_{12}^p(\tau_k^*)$, bearing in mind that for eqns (11), (9), (29) and (99) we have

$$e_{12}^p(\tau_k^*) = (1+\eta)^{-1}(\xi \sin(\tau_k^*) - \chi_2(\tau_k^*)). \tag{108}$$

The equations that have just been found make it possible to calculate $\zeta(\tau_k^*)$ for each $k > 1$ and, therefore, the first term on the right-hand side of eqn (104); we shall now go on to calculate the second term. In view of eqn (27), we have

$$\|\mathbf{E}^p(\tau_k^*)\|^2 = \tfrac{3}{2}e_{11}^p(\tau_k^*)^2 + 2e_{12}^p(\tau_k^*)^2\,; \tag{109}$$

to begin with, we see that in eqn (104) the quantity $\|\mathbf{E}^p(\tau_k^*)\|^2$ is found to be multiplied by η, so that in order to calculate the internal work within an error of the order $\sigma(\eta)$, we can use eqns (77) and (79) deduced for an ideally plastic material; also, while $|e_{11}^p(\tau)|$ is an increasing function of τ, it follows from eqn (108) that $e_{12}^p(\tau_k^*)$ oscillates around zero, so that, for a fairly large number of cycles, its contribution in the calculation of $\|\mathbf{E}^p(\tau_k^*)\|^2$ can be disregarded. Finally, we can write

$$\mu\eta \| \mathbf{E}^{\mathrm{p}}(\tau_k^*) \|^2 \simeq \tfrac{3}{2}\mu\eta e_{11}^{\mathrm{p}}(\tau_k^*)^2, \tag{110}$$

where, for eqns (79) and (107),

$$e_{11}^{\mathrm{p}}(\tau_k^*) \simeq e_{11}^{\mathrm{p}}(\tau_{k-1}^*) + \tfrac{1}{3}(f/\mu)\rho_0^{-1}\zeta_{\mathrm{p}}(\tau_k^*), \tag{111}$$

with the first term $e_{11}^{\mathrm{p}}(\tau_1^*)$ calculated from eqn (79).

The procedure for the computation of the internal work proves to be somewhat laborious because, for each k, the calculation of the instant τ_k of the kth yielding requires knowledge of $e_{12}^{\mathrm{p}}(\tau_{k-1}^*)$. It can therefore be of some use in applications to follow a method that makes it possible to give an estimate from below an upper bound of the internal work without knowing the values of τ_k for $k > 1$. For this purpose, we see that with the help of eqns (103) and (108), we obtain

$$1+(-1)^{k-1}\sin(\tau_{k-1}) \geqslant 1+(-1)^k \sin(\tau_k) \geqslant 1+s_\infty, \tag{112}$$

where s_∞ is the limit of the sequence $(-1)^k \sin(\tau_k)$ for $k \to \infty$, which corresponds to the situation in which $\chi_1 = 0$ (Fig. 3). In view of eqns (26) and (99), we find that

$$s_\infty = 1-\sqrt{2}\rho_0\xi^{-1}. \tag{113}$$

From eqns (106), (112) and (113) it follows that

$$\zeta(\tau_k^*) \geqslant \zeta(\tau_1^*) + (k-1)\zeta_\infty\{1-\eta\xi^{-2}[1+\tfrac{1}{2}\rho_0^{-1}(1-\xi^2)\zeta_\infty]\}, \tag{114}$$

where

$$\zeta_\infty = 2\rho_0(1-s_\infty)^{-1}(1+s_\infty) = 2(\sqrt{2}\xi-\rho_0). \tag{115}$$

From eqns (114), (115), (63) and (26) it follows that

$$\zeta(\tau_k^*) \geqslant (2k-1)(\sqrt{2}\xi-\rho_0)\{1-\eta\xi^{-2}[1+\rho_0^{-1}(1-\xi^2)(\sqrt{2}\xi-\rho_0)]\}. \tag{116}$$

An estimate can be deduced for $\| \mathbf{E}^{\mathrm{p}}(\tau_k^*) \|$ too. To begin with, we see that ζ_∞ coincides with the value of ζ at instant τ_1^* in the case in which $f = 0$, and therefore for each k, we have

$$\zeta_\infty \leqslant \zeta_{\mathrm{p}}(\tau_k^*). \tag{117}$$

Moreover, for each k, $e_{11}^{\mathrm{p}}(\tau_k^*)$ has the same sign as f, so that in view of eqns (112), (115), (111) and (117), we have

$$|e_{11}^{\mathrm{p}}(\tau_k^*)| \geqslant |e_{11}^{\mathrm{p}}(\tau_{k-1}^*)| + \tfrac{1}{3}(|f|/\mu)\rho_0^{-1}\zeta_\infty. \tag{118}$$

From this last inequality, taking into account eqns (79) and (115), we obtain

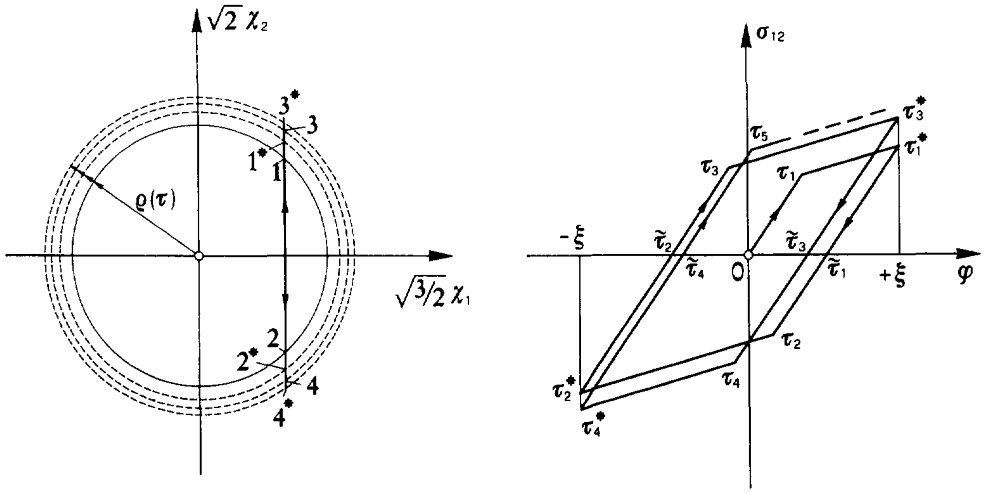

Fig. 4. Deformation process and hysteresis diagram for isotropic hardening.

$$|e^{p}_{11}(\tau^*_n)| > (2n-1)|f|(3\mu\rho_0)^{-1}(\sqrt{2}\xi-\rho_0). \tag{119}$$

from which we immediately obtain the required estimation of $\|\mathbf{E}^p(\tau^*_n)\|$. In the case of isotropic hardening (Fig. 4), the internal work after k oscillations is, for eqn (16),

$$w[0,\tilde{\tau}_k] = 2\mu\omega(\zeta(\tau^*_k)), \tag{120}$$

where $\omega(\zeta(\tau^*_k))$ can be calculated with the help of eqn (97). We thus get

$$\omega(\zeta(\tau^*_k)) \simeq \rho_0 \sum_{n=1}^{k} \zeta_p(\tau^*_n)q_n(\tau^*_n), \tag{121}$$

where $\zeta_p(\tau^*_n)$ can be calculated from eqn (106), since the instants of yielding, τ_k, are known. These instants can be easily obtained from eqn (103), with the help of eqns (93) and (94).

In this case, we cannot obtain an estimate of the internal work in the same way as the one obtained in the case of kinematic hardening, because as the number of cycles increases, the radius of the elastic range also increases up to a point at which further plastic loading situations are prevented, as can be seen in Fig. 4.

Lastly, let us examine the case of an ideally plastic material (Fig. 5). In view of eqns (75) and (16), we have

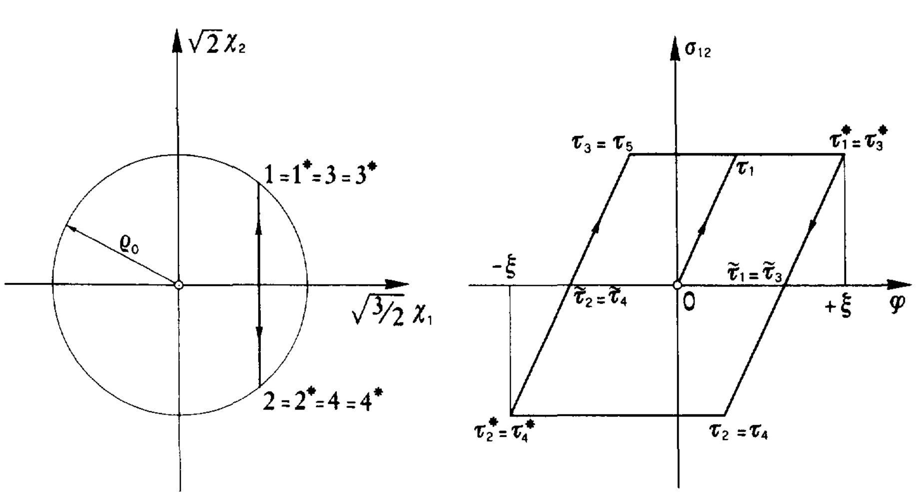

Fig. 5. Deformation process and hysteresis diagram for ideal plasticity.

$$w[0, \tilde{\tau}_k] = 2\mu\rho_0(\zeta(\tau_1^*) + \sum_{n=2}^{k} \zeta_p(\tau_n^*)). \tag{122}$$

We see that, during each plastic loading phase, χ_2 is constant, as can be easily deduced from Fig. 5, so that

$$\tau_k = \pi + \tau_{k-1}. \tag{123}$$

Moreover, in view of eqns (101) and (102), we have

$$\chi_2(\tau_k) = (-1)^k \xi \sin(\tau_1), \tag{124}$$

$$\varphi(\tau_k) = (-1)^k \xi(1 - 2\sin\tau_1), \tag{125}$$

and these, together with eqn (100), imply that for $k > 1$

$$\zeta_p(\tau_k^*) = 2\rho_0(\sin\tau_1)^{-1}(1 - \sin\tau_1), \tag{126}$$

where, in view of eqns (26) and (99),

$$\sin(\tau_1) = \xi^{-1}\sqrt{\tfrac{1}{2}\rho_0^2 - \tfrac{1}{3}(f/2\mu)^2}. \tag{127}$$

From eqns (126) and (127), the following expression is obtained for the internal work:

$$w[0, \tilde{\tau}_k] = 2\mu(2k-1)\rho_0^2\{2\sqrt{3}\xi\mu(\sqrt{6\rho_0^2\mu^2 - f^2})^{-1} - 1\}. \tag{128}$$

Moreover, we deduce from eqns (111) and (126)

$$e_{11}^{p}(\tau_k^*) = \tfrac{1}{3}(f/\mu)(2k-1)(\sin\tau_1)^{-1}(1 - \sin\tau_1) \tag{129}$$

and therefore the longitudinal displacement per unit length, in view of eqn (127), is

$$e_{11}(\tau_k^*) = \tfrac{1}{3}(f/\mu)\{1 + (2k-1)[2\sqrt{3}\xi\mu(\sqrt{6\rho_0^2\mu^2 - f^2})^{-1} - 1]\}. \tag{130}$$

In order to have a prompt understanding of the behaviour of the internal work and the longitudinal plastic deformation, we define the following adimensional quantities:

$$W_k = w/[2\mu(2k-1)\rho_0^2], \quad E_k^{p} = e_{11}^{p}/[\tfrac{1}{3}(2k-1)(f/\mu)], \tag{131}$$

$$F = f/[2\sqrt{3}\rho_0\mu], \quad G = \sqrt{2}\xi/\rho_0, \tag{132}$$

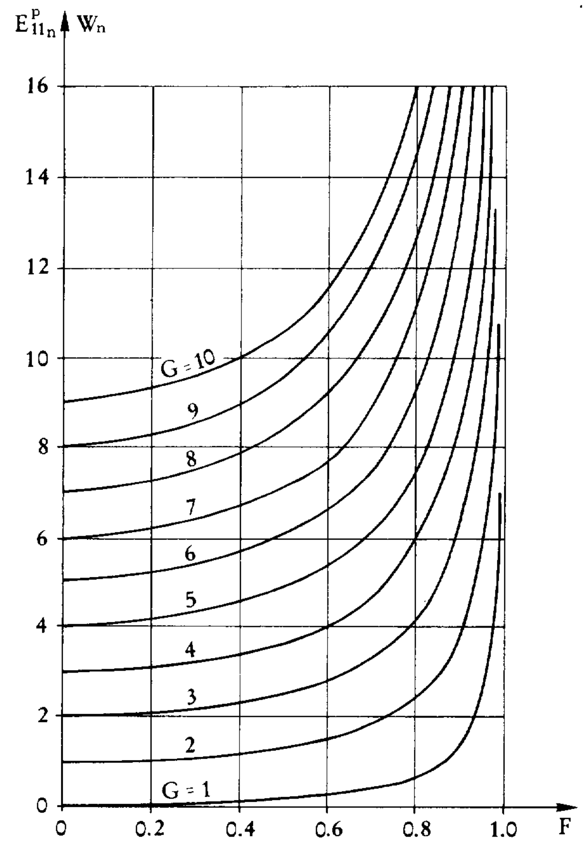

Fig. 6. Adimensional curves of $W_k(F, G)$ and $E^p_k(F, G)$.

from which it is easy to obtain the family of curves

$$W_k = E^p_k = G(1-F^2)^{-1/2} - 1, \tag{133}$$

represented in Fig. 6.

In applications, we know the values of F and G as functions of the material constants ρ_0, μ and the loads f, ξ. Thus, we can calculate the internal work and the longitudinal plastic deformation for each cycle with the help of relationships (131) and the curves in Fig. 4.

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